An impure public good model with lotteries in large groups*

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Abstract

We analyze the effect of a large group on an impure public goods model with lotteries. We show that as populations get large, and with selfish preferences, the level of contributions converges to the one given by voluntary contributions. With altruistic preferences (of the *warm glow* type), the contributions converge to a level strictly higher than those given by voluntary contributions, even though in general they do not yield first-best levels.

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1 Introduction

Most public goods in modern economies are provided by the government and funded from revenues obtained via general taxes. But coercitive taxation has its limits, for reasons that have to do both with the inefficient (or second-best) way in which it is collected, and for political economy reasons. Yet, some legitimate needs are not covered by the ordinary revenues from the state, and both private and public entities resort to other mechanisms to fund those public goods. As it is well known (e.g. Bergstrom, Blume, and Varian (1986)), providing them via voluntary contributions usually leads to inefficient outcomes, so it is not surprising that human ingenuity has devised other means to achieve the goal of providing public goods efficiently.

One such method is a lottery in which a prize is given to the winner(s), but a fraction of the proceeds goes to the provision of public goods. For a while, there was a theoretical controversy about the usefulness of lotteries to improve efficiency (see e.g. Borg, Mason, and Shapiro (1991)) or equity (see e.g. Clotfelter and Cook (1989)) in public goods provision. This was essentially settled when Morgan (2000) showed that lotteries can be used effectively to solve the problem. He proved that lotteries significantly increase the level of contributions above the one given by voluntary contributions. He also showed that for large enough prizes, the lottery could make the provision of the public good reach first-best levels.

The aim of this note is to establish the limits to the usefulness of lotteries in the provision of public goods. We show that as populations get large, and with standard preferences (for which an individual only cares about his own material well-being), the level of contributions converges to the (inefficient) one given by voluntary contributions. A more positive result arises when one considers people with altruistic preferences as in the *warm glow* of giving model of Andreoni (1989, 1990). In large populations, when people have these preferences the contributions converge to a level strictly higher than those given by voluntary contributions (still under *warm glow* preferences), even though in general they do not yield first-best levels.

Our results clarify why it is so important that lottery proceeds are earmarked to *worthy* causes, where *warm glow* is likely to be larger. In this way we shed light on a controversy about the meaningfulness of earmarking (see e.g.

Buchanan (1963) and Borg and Mason (1988)) because of the fungible nature of government revenues. They also explain why in general governments do not rely on lotteries for a large part of the revenue creation for public good provision.

Section 2 describes the reference benchmark model from Morgan (2000) and also introduces *warm glow* preferences into such model. Section 3 presents the results for large populations. Section 4 briefly concludes.

2 The Reference Model

We first recapitulate the results of Morgan (2000). He shows that his results for the provision of public goods by means of lotteries also apply in the more general case analyzed by Bergstrom and Cornes (1983), who provide a specification of preferences in which income effects are present and public goods allocation decision is separate from distributional decisions. They argue that this is essentially equivalent to assuming that individual preferences can be represented as a quasi-concave utility function of the form[1],

$$U_i^{FB} = \omega_i H(G) + h_i(G),$$

where $H(\cdot) > 0$.

For the first-best benchmark, the optimal public good provision, which we denote by G^* solves

$$\max_{G \in \mathcal{R}^+} \sum_{i=1}^{n} \left(\omega_i H(G) + h_i(G) \right). \tag{1}$$

The individual preferences when the public good is provided by voluntary contributions are:

$$U_i^{VC} = (\omega_i - x_i) H(G) + h_i(G). \tag{2}$$

Let $\widehat{x} \equiv \sum_{i=1}^{n} x_i$ and since contributions pay for the public good $G = \widehat{x}$. The provision of public good by voluntary contributions, denoted by G^V is the equilibrium of the game in which each agent maximizes U_i^{VC} with respect to his contribution x_i noting that $G = \widehat{x}$.

[1] Bergstrom and Cornes (1983) also provide a recipe for constructing quasi-concave functions of this form and a diagnostic test to determine whether a given function of this form is quasi-concave.

Finally, in the lottery model of Morgan (2000), the utility function of agent i takes into account that $x_i/\widehat{x}$ represents the probability that individual i wins the prize. Since the sum of all wagers must pay for the prize R, the public good provision, denoted by G^L, is determined by the excess of wagers over the prize, that is:

$$G^L = \widehat{x} - R$$

In this case the utility of agent i is:

$$U_i^L = \left(\omega_i - x_i + R\frac{x_i}{\widehat{x}}\right) H(\widehat{x} - R) + h_i(\widehat{x} - R).$$

In this case the provision of public good using the lottery scheme, G^L, is the equilibrium of the game in which each agent maximizes U_i^L. For simplicity of exposition we will assume that for all games Γ we will describe in what follows

Assumption 1 U_i^Γ *satisfies:*

1. *It is twice continuously differentiable and concave in the decision variable* x_i
2. $\left.\frac{\partial U_i^\Gamma}{\partial x_i}\right|_{x_i=0, x_j=\omega_j} > 0$, $\left.\frac{\partial U_i^\Gamma}{\partial x_i}\right|_{x_i=\omega_i} < 0$

Using 1. in assumption 1 we can characterize equilibria using first order conditions, and using 2. we guarantee solutions are interior. We can now show that:

Proposition 2 G^V, G^* *and* G^L *satisfy:*

1. $G^V < G^*$.
2. $G^V < G^L$.
3. $G^L \to G^*$ *as* $R \to \infty$.

Proof. See the Appendix. ■

2.1 Incorporating the warm glow of giving in the reference model

We now incorporate the *warm glow* approach of Andreoni (1989, 1990) into the reference model by assuming that individual preferences can be represented as follows,

$$U_i^{wg} = (\omega_i - x_i)H(G) + h_i(G)g(x_i), \tag{3}$$

where the function $g(\cdot)$ represents the *warm glow of giving*. Setting

$$g(x_i) = g_1 x_i + g_0$$

the particular case where $g_1 = 0$ and $g_0 = 1$ corresponds to (2), the model used by Morgan (2000) to show the robustness of his results.

The provision of public good by voluntary contributions when preferences are as in (3), denoted by G^{wg}, is the equilibrium of the game in which each agent maximizes U_i^{wg}. For this game we can show that:

Proposition 3 $G^V < G^{wg}$

Proof. See the appendix. ■

In Temimi (2001) the author shows that the introduction of warm-glow affects both the equilibrium level as well as the efficient level of public good provision. The condition determining the efficient level of provision for public good case requires as usual that the sum of the marginal rates of substitution (between the public good and the net private good) is equal to one. In our case this is true when,

$$\sum_{i=1}^{n} \frac{\frac{\partial U_i^{wg}}{\partial G}}{\frac{\partial U_i^{wg}}{\partial(\omega_i - x_i)} - \frac{\partial U_i^{wg}}{\partial x_i}} = 1$$

Applied to the model in (3), the efficient level of public good provision under warm-glow, G^{wg*}, is given by the solution to:

$$\sum_{i=1}^{n} \frac{(\omega_i - x_i)H'(G^{wg*}) + g(x_i)h_i'(G^{wg*})}{H(G^{wg*}) - h_i(G^{wg*})g_1} = 1 \tag{4}$$

2.2 Lottery in the warm-glow model and the efficient level of contributions

Now we incorporate the lottery mechanism of Morgan (2000) into the above model of *warm glow*. Individual i now chooses x_i to maximize

$$U_i^{wgL} = \left(\omega_i - x_i + R\frac{x_i}{\widehat{x}}\right) H(\widehat{x} - R) + h_i(\widehat{x} - R)g(x_i) \tag{5}$$

As before, wagers pay for the prize R, so the public good provision, denoted by G^{wgL}, is:

$$G^{wgL} = \widehat{x} - R$$

Proposition 4 *1.* $G^{wg} < G^{wgL}$.

2. When $h_i(\cdot) = h(\cdot)$, $G^{wgL} \to G^{wg*}$ *as* $R \to \infty \Longleftrightarrow g_1 = 0$.

Proof. See the appendix. ∎

3 The case of large populations

One result in Morgan (2000) shows that wagers in the unique equilibrium provide levels of public good close to first-best as the lottery prize increases. However, we have shown that this does not hold when $g_1 \neq 0$ (i.e. with warm glow) for the impure public good case, at least when agents are homogenous. And even when $g_1 = 0$, if we allow the prize to reach arbitrarily large sizes, the prize R will eventually be greater than $n\omega$, the maximum aggregate bid for given n and ω. However, R is only useful if chosen so that in an interior symmetric equilibrium the level of provision $nx - R$ is positive (where x is the contribution for each person). That is, a lottery prize yielding social benefits in terms of the public good must have $R = n\rho$ with $x > \rho$.

In what follows we analyze the effect of increasing the prize in proportion to the group size with homogeneous agents.

3.1 The linear case with identical agents

In order to illustrate the main point, let us first see what happens when the utility function of agent i is as in (3) and $H(\cdot)$ and $h(\cdot)$ are increasing and linear functions.

Proposition 5 *Suppose* $R = n\rho$ *and* $H(\cdot)$ *and* $h(\cdot)$ *are increasing and linear functions with* $H - hg_1 > 0$. *Then for any symmetric equilibrium where the individual contribution* x *satisfiess* $\rho < x$, *we have that* $\limsup_{n \div \infty} \left| G^{wg*} - G^{wgL} \right| / n > 0$.

Proof. See the appendix. ∎

3.2 A more general model

Now, let us consider a general case in which $H(\cdot)$ and $h(\cdot)$ are general increasing, differentiable and strictly concave functions. That is, $H'(\cdot) > 0$, $h'(\cdot) > 0$ and

$H''(\cdot) < 0$, $h''(\cdot) < 0$. With $H'(\cdot) - h'(\cdot)g_1 > 0$. The function $g(\cdot)$ remains a linear function. Now we have:

Proposition 6 $\frac{\partial G^{wg}}{\partial n} > 0$.

Proof. See the appendix. ■

Remark 7 *From the proof of Proposition 6 one can see that under quasi-linear preferences with $H(\cdot) \equiv 1$ and in the absence of* warm glow *(as in one of the benchmark models of Morgan (2000)) the provision of the public group is invariant with respect to n.*

If we introduce lotteries in the proposed model, we obtain that:

$$U_i^{wgL} = \left(\omega_i - x_i + R\frac{x_i}{\widehat{x}}\right) H(\widehat{x} - R) + h(\widehat{x} - R)g(x_i)$$

Proposition 8 *Suppose $R = n\rho$ and $\rho < x$ for any symmetric equilibrium with individual contribution x. Then if $H(\cdot)$ and $h(\cdot)$ are such that*

$\lim_{y\to\infty} H(y)/h(y) = k$, *and* $1 > kg_1$ *in a symmetric equilibrium*

$$\lim_{n\to\infty} x = \frac{\rho}{1 - kg_1}$$

Proof. See the appendix. ■

As a result, if $k = 0$ or $g_1 = 0$ then x approaches the corner solution $x = \rho$.

4 Conclusions

In this note we have shown that lotteries have limits as a tool to achieve efficient public good provision in large populations. But we also show that lotteries are clearly more effective than voluntary contributions when individuals experience a *warm glow* of giving to public goods. One concrete empirical implication from our analysis is that goods likely to produce a *warm glow* are more likely financed in this way. This could be useful to analyze empirically the extent to which the effects characterized in this paper are present in the field.

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5 Appendix

Proof of Proposition 2

Proof. This proposition is already shown in Morgan (2000), we merely add it here for completeness.

G^* solves

$$\sum_{i=1}^{n} h_i'(G^*) = H(G^*) - H'(G^*)\left(\sum_{i=1}^{n} \omega_i - G^*\right). \tag{6}$$

and we also obtain G^V by adding first-order conditions of optimization problems for each agent i,

$$\sum_{i=1}^{n} h_i'(G^V) = nH(G^V) - H'(G^V)\left(\sum_{i=1}^{n} \omega_i - G^V\right). \tag{7}$$

It is then easy to verify that $G^V < G^*$. Also G^L, solves the sum of first-order conditions.

$$\sum_{i=1}^{n} h_i'(G^L) = H(G^L)\left(n - (n-1)\frac{R}{R+G^L}\right) - H'(G^L)\left(\sum_{i=1}^{n} \omega_i - G^L\right) \tag{8}$$

Comparing expressions (7) and (8), Morgan (2000) shows that $G^V < G^L$, and that as $R \to \infty$, expression (8) becomes identical to (6). ■

Proof of Proposition 3

Proof. At an interior maximum, the first-order condition of (3) with respect to x,

$$-H(G) + (\omega_i - x_i)H'(G) + h_i'(G)(g_1 x_i + g_0) + h_i(G)g_1 = 0 \tag{9}$$

The equilibrium level of public good provided by voluntary contributions with the presence of *warm glow* giving, solves the sum of first-order conditions,

$$\begin{aligned}
\sum_{i=1}^{n} h_i'(G^{wg})(g_1 x_i + g_0) + \sum_{i=1}^{n} h_i(G^{wg})g_1 &= nH(G^{wg}) \\
&- H'(G^{wg})(\sum_{i=1}^{n} \omega_i - G^{wg})
\end{aligned}$$

Then, we have

$$\begin{aligned}
\sum_{i=1}^{n} h_i'(G^{wg})g_0 &= nH(G^{wg}) - H'(G^{wg})(\sum_{i=1}^{n} \omega_i - G^{wg}) \\
&- \sum_{i=1}^{n} h_i'(G^{wg})g_1 x_i - \sum_{i=1}^{n} h_i(G^{wg})g_1
\end{aligned} \tag{10}$$

Without loss of generality, set $g_0 = 1$ and compare expressions (7) and (10) to verify that the result holds. ■

Proof of Proposition 4

Proof. Take the first-order conditions of (5) with respect to x_i to find

$$\begin{aligned}\left(R\frac{\widehat{x}-x_i}{\widehat{x}^2}-1\right)H(\widehat{x}-R)+\left(\omega_i-x_i+R\frac{x_i}{\widehat{x}}\right)H'(\widehat{x}-R) \quad &+ \\ h_i'(\widehat{x}-R)(g_1x_i+g_0)+h_i(\widehat{x}-R)g_1 \quad &= \quad 0\end{aligned}$$

The public goods provision G^{wgL} solves the sum of the first-order conditions,

$$\begin{aligned}\sum_{i=1}^n h_i'(G^{wgL})(g_1x_i+g_0)+\sum_{i=1}^n h_i(G^{wgL})g_1 \quad &= \quad H(G^{wgL})\left(n-(n-1)\frac{R}{R+G^{wgL}}\right) \\ &\quad -H'(G^{wgL})\left(\sum_{i=1}^n \omega_i - G^{wgL}\right)\end{aligned}$$

We have

$$\begin{aligned}\sum_{i=1}^n h_i'(G^{wgL})g_0 \quad &= \quad H(G^{wgL})\left(n-(n-1)\frac{R}{R+G^{wgL}}\right)-H'(G^{wgL})\left(\sum_{i=1}^n \omega_i - G^{wgL}\right) \\ &\quad -\sum_{i=1}^n h_i'(G^{wgL})g_1x_i-\sum_{i=1}^n h_i(G^{wgL})g_1\end{aligned} \tag{11}$$

Notice that expressions (10) and (11) differ by the term associated to the negative externality of the lottery multiplied by $H(G)$. Similar to the model without warm-glow, the public goods provision under the lottery is greater than under voluntary contributions. That is, $G^{wg} < G^{wgL}$.

When $R \to \infty$, we obtain the expression

$$\begin{aligned}\sum_{i=1}^n h_i'(G^{wgL})g_0 \quad &= \quad H(G^{wgL})-H'(G^{wgL})\left(\sum_{i=1}^n \omega_i - G^{wgL}\right) \\ &\quad -\sum_{i=1}^n h_i'(G^{wgL})g_1x_i-\sum_{i=1}^n h_i(G^{wgL})g_1\end{aligned} \tag{12}$$

From (4), we have that,

$$\begin{aligned}\sum_{i=1}^n \frac{h_i'(G^{wg*})}{H(G^{wg*})-h_i(G^{wg*})g_1}g_0 \quad &= \quad 1-H'(G^{wg*})\sum_{i=1}^n \frac{\omega_i-x_i}{H(G^{wg*})-h_i(G^{wg*})g_1} \\ &\quad - \quad g_1\sum_{i=1}^n \frac{x_ih_i'(G^{wg*})}{H(G^{wg*})-h_i(G^{wg*})g_1}\end{aligned} \tag{13}$$

When $h_i(\cdot) = h(\cdot)$ and $g_1 \neq 0$, expressions (12) and (13) respectively reduce to:

$$\begin{aligned} nh'(G^{wgL})g_0 &= H(G^{wgL}) - H'(G^{wgL}) \left(\sum_{i=1}^{n} \omega_i - G^{wgL} \right) \\ &- g_1 h'(G^{wgL})G^{wgL} - ng_1 h(G^{wgL}) \end{aligned}$$

$$\begin{aligned} nh'(G^{wg*})g_0 &= H(G^{wg*}) - H'(G^{wg*}) \left(\sum_{i=1}^{n} \omega_i - G^{wg*} \right) \\ &- g_1 h'(G^{wg*})G^{wg*} - g_1 h(G^{wg*}) \end{aligned}$$

It is easy to verify that G^{wgL} does not converge to G^{wg*} as $R \to \infty$. However, as we have already seen when $g_1 = 0$, the result is identical to Morgan (2000) in which $G^L \to G^*$ as $R \to \infty$. ■

Proof of Proposition 5

Proof. Solving (13) to obtain the optimal level of provision, we have

$$G^{wg*} = n \frac{H\omega + hg_0}{2(H - hg_1)}$$

The first-order condition of (5) with respect to x assuming symmetry and $R = n\rho$ is,

$$\left(n\rho \frac{n-1}{n^2 x} - 1 \right) Hn(x-\rho) + \left(\omega - x + n\rho \frac{1}{n} \right) H + hg_1 x + hg_0 + hn(x-\rho)g_1 = 0$$

Hence,

$$\left(n\rho \frac{n-1}{n^2 x} - 1 \right) Hn(x-\rho) + (\omega - x + \rho)H + hg_1 x + hg_0 + hn(x-\rho)g_1 = 0$$

$$\rho \frac{n-1}{x} H(x-\rho) - Hn(x-\rho) + (\omega - x + \rho)H + hg_1 x + hg_0 + hn(x-\rho)g_1 = 0$$

dividing by n and letting $n \to \infty$

$$\rho H(x-\rho)\frac{1}{x} - H(x-\rho) + h(x-\rho)g_1 = 0$$

$$(x-\rho)\left(\rho H \frac{1}{x} - H + hg_1 \right) = 0$$

This equation has two solutions, one is $x_1 = \rho$. But since we require that $x - \rho > 0$, it is not a valid one. The other solution solves

$$\rho H \frac{1}{x} - H + hg_1 = 0$$

Then

$$x_2 = \frac{H}{H - hg_1}\rho$$

where $x_2 > 0$ since $H - hg_1 > 0$. This is an interior solution since $x_2 > \rho$. In this case, the level of public good is:

$$G^{wgL} = nx_2 - R = nx_2 - n\rho = n\frac{hg_1}{H - hg_1}\rho.$$

As a result, we can verify that

$$\frac{G^{wg*} - G^{wgL}(x_2)}{n} = \frac{H\omega + hg_0}{2(H - hg_1)} - \frac{hg_1}{H - hg_1}\rho$$

Now

$$\limsup_{n\to\infty} \frac{G^{wg*} - G^{wgL}(x_2)}{n} = \frac{H\omega + hg_0}{2(H - hg_1)} - \frac{hg_1}{H - hg_1}\rho$$

Hence in the case where

$$\frac{H\omega + hg_0}{2(H - hg_1)} - \frac{hg_1}{H - hg_1}\rho \neq 0$$

the result follows. If

$$\frac{H\omega + hg_0}{2(H - hg_1)} - \frac{hg_1}{H - hg_1}\rho = 0$$

then the inequality

$$\begin{aligned} x_2 - \omega &= \frac{H}{H - hg_1}\frac{H\omega + hg_0}{2hg_1} - \omega \\ &= \omega\frac{(H - hg_1)^2 + (hg_1)^2}{(H - hg_1)\,2hg_1} + \frac{Hhg_1}{(H - hg_1)\,2hg_1} > 0 \end{aligned}$$

which contradicts $x < \omega$ and the result follows. ■

Proof of Proposition 6

Proof. As we have shown in section 2.1, the first order condition of (3) is

$$-H(G) + (\omega_i - x_i)H'(G) + h_i'(G)(g_1 x_i + g_0) + h_i(G)g_1 = 0$$

For the symmetric case,

$$-H(G) + (\omega - x)H'(G) + h'(G)(g_1 x + g_0) + h(G)g_1 = 0$$

Comparative statics

$$\begin{aligned} -H'(G)\frac{\partial G}{\partial n} + (\omega - x)H''(G)\frac{\partial G}{\partial n} - \frac{\partial x}{\partial n}H'(G) &\quad + \\ h''(G)\frac{\partial G}{\partial n}(g_1 x + g_0) + h'(G)g_1\frac{\partial x}{\partial n} + h'(G)\frac{\partial G}{\partial n}g_1 &= 0 \end{aligned}$$

Then

$$\frac{\partial G}{\partial n}[(\omega - x)H''(G) + h''(G)(g_1 x + g_0) - H'(G) + h'(G)g_1] = \frac{\partial x}{\partial n}[H'(G) - h'(G)g_1]$$

We have that

$$(\omega - x)H''(G) + h''(G)(g_1 x + g_0) - H'(G) + h'(G)g_1 < 0$$

and

$$H'(G) - h'(G)g_1 > 0$$

then

$$sign\frac{\partial G}{\partial n} \neq sign\frac{\partial x}{\partial n}$$

Since $\frac{\partial G^{wg}}{\partial n} = \frac{\partial (nx)}{\partial n} = x + n\frac{\partial x}{\partial n}$. Then, $\frac{\partial x}{\partial n} < 0$ and $\frac{\partial G^{wg}}{\partial n} > 0$. ■

Proof of Proposition 8

Proof. The first-order condition,

$$\begin{aligned}\left(R\frac{\widehat{x} - x_i}{\widehat{x}^2} - 1\right) H(\widehat{x} - R) + \left(\omega_i - x_i + R\frac{x_i}{\widehat{x}}\right) H'(\widehat{x} - R) \quad + \\ h_i'(\widehat{x} - R)(g_1 x_i + g_0) + h_i(\widehat{x} - R)g_1 \quad = \quad 0\end{aligned}$$

where $\widehat{x} = \sum x_i$ and by symmetry and assuming that $R = n\rho$, we have

$$\begin{aligned}\left(n\rho\frac{n-1}{n^2 x} - 1\right) H(nx - n\rho) + \left(\omega - x + n\rho\frac{1}{n}\right) H'(nx - n\rho) \quad + \\ h'(nx - n\rho)(g_1 x + g_0) + h(nx - n\rho)g_1 \quad = \quad 0\end{aligned}$$

This is equivalent to

$$\begin{aligned}\left(\rho\frac{n-1}{nx} - 1\right) H(nx - n\rho) + (\omega - x + \rho)H'(nx - n\rho) \quad + \\ h'(nx - n\rho)(g_1 x + g_0) + h(nx - n\rho)g_1 \quad = \quad 0\end{aligned} \tag{14}$$

Returning to (14) and letting H' and h' tend to 0 as $n \to \infty$, we obtain

$$\rho H(nx - n\rho)\frac{1}{x} - H(nx - n\rho) + h(nx - n\rho)g_1 = 0$$

That is,

$$\rho H(nx - n\rho) - x[H(nx - n\rho) - h(nx - n\rho)g_1] = 0$$

$$x = \frac{\rho H(nx - n\rho)}{H(nx - n\rho) - h(nx - n\rho)g_1}$$

Since $H' - h'g_1 > 0$ then $H - hg_1 > 0$. Rewriting,

$$x = \frac{\rho}{1 - \frac{h(nx-n\rho)}{H(nx-n\rho)} g_1}$$

If we assume that $\frac{h(nx-n\rho)}{H(nx-n\rho)} \to k$ as $n \to \infty$ with $k > 0$, then the unique interior equilibrium can be written as,

$$x = \frac{\rho}{1 - kg_1}$$

■